FP1 JUNE 2011

1.
$$f(x) = 3^x + 3x - 7$$

(a) Show that the equation $f(x) = 0$ has a root α between $x = 1$ and $x = 2$.
(2)

(b) Starting with the interval $[1, 2]$, use interval bisection twice to find an interva[l of] width 0.25 which contains α.
(3)

a) $f(1) = -1$ change of sign $\Rightarrow$ $1 < \alpha < 2$

$f(2) = 8$ $\alpha \in (1, 2)$

b)

1	1.25	1.5	1.75	2
$f(1)$	$f(1.25)$	$f(1.5)$		$f(2)$
-1	0.698	2.696		8

$\longleftarrow \alpha \longrightarrow$

$\alpha \in (1, 1.25)$ $1 < \alpha < 1.25$

2. $$z_1 = -2 + \mathrm{i}$$

(a) Find the modulus of z_1.

(b) Find, in radians, the argument of z_1, giving your answer to 2 decimal places.

The solutions to the quadratic equation

$$z^2 - 10z + 28 = 0$$

are z_2 and z_3.

(c) Find z_2 and z_3, giving your answers in the form $p \pm \mathrm{i}\sqrt{q}$, where p and q are integers. **(3)**

(d) Show, on an Argand diagram, the points representing your complex numbers z_1, z_2 and z_3. **(2)**

a) $|z_1| = |-2+i| = \sqrt{2^2+1^2} = \sqrt{5}$

b) $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sqrt{5}\times 1}{\sqrt{2}\times -2} \Rightarrow \theta = \tan^{-1}\left(\frac{-1}{2}\right) = -0.4636..$

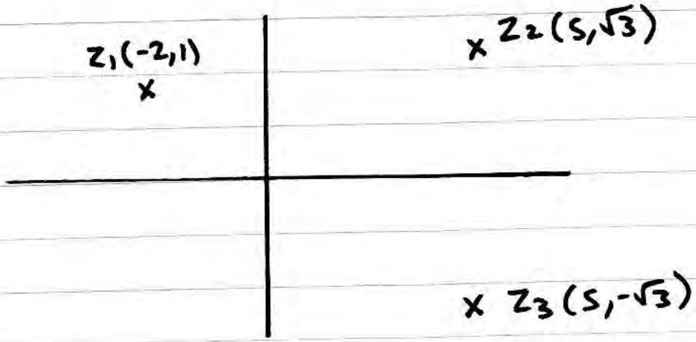
$+\pi$ $\quad \arg(z_1) = \underline{2.68^c}$

c) $(z-5)^2 - 25 + 28 = 0$

$\Rightarrow (z-5)^2 = -3 \quad \Rightarrow \quad z - 5 = \pm\sqrt{-3} \quad \Rightarrow z = 5 \pm i\sqrt{3}$

$\therefore \quad z_2 = 5 + i\sqrt{3} \qquad z_3 = 5 - i\sqrt{3}$

d)



3. (a) Given that

$$\mathbf{A} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$$

(i) find $\mathbf{A}^2$,

(ii) describe fully the geometrical transformation represented by $\mathbf{A}^2$.

(4)

(b) Given that

$$\mathbf{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

describe fully the geometrical transformation represented by **B**.

(2)

(c) Given that

$$\mathbf{C} = \begin{pmatrix} k+1 & 12 \\ k & 9 \end{pmatrix}$$

where k is a constant, find the value of k for which the matrix **C** is singular.

(3)

i) $A^2 = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

ii) englargement s.f 3 centre origin.

b) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ reflection through line $y=-x$

c) $\det C = 9(k+1) - 12k = 9 - 3k$

$\det C = 0$ if singular $\therefore$ $\underline{k=3}$

4. $$f(x) = x^2 + \frac{5}{2x} - 3x - 1, \quad x \neq 0$$

(a) Use differentiation to find $f'(x)$.

(2)

The root α of the equation $f(x) = 0$ lies in the interval $[0.7, 0.9]$.

(b) Taking 0.8 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places.

(4)

a) $f(x) = x^2 + \frac{5}{2}x^{-1} - 3x - 1$

$f'(x) = 2x - \frac{5}{2}x^{-2} - 3$

$x_1 = 0.8 \quad x_2 = 0.8 - \frac{f(0.8)}{f'(0.8)} = 0.8 - \frac{\frac{73}{200}}{-\frac{849}{160}}$

$x_2 \approx \underline{0.869}$ (3dp)

5. $$\mathbf{A} = \begin{pmatrix} -4 & a \\ b & -2 \end{pmatrix},$$ where a and b are constants.

Given that the matrix **A** maps the point with coordinates $(4, 6)$ onto the point with coordinates $(2, -8)$,

(a) find the value of a and the value of b.

(4)

A quadrilateral R has area 30 square units.
It is transformed into another quadrilateral S by the matrix **A**.
Using your values of a and b,

(b) find the area of quadrilateral S.

(4)

a) $\begin{pmatrix} -4 & a \\ b & -2 \end{pmatrix}\begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \end{pmatrix} \Rightarrow \begin{matrix} -16 + 6a = 2 \Rightarrow a = 3 \\ 4b - 12 = -8 \Rightarrow b = 1 \end{matrix}$

b) $\begin{pmatrix} -4 & 3 \\ 1 & -2 \end{pmatrix}$ $\det A = 8 - 3 = 5$

$\text{Area } S = 5 \times 30 = \underline{150}$ square units

6. Given that $z = x + iy$, find the value of x and the value of y such that

$$z + 3i z^* = -1 + 13i$$

where z^* is the complex conjugate of z.

$(x+iy) + 3i(x-iy) = x + iy + i3x - 3i^2 y$

$= (x+3y) + (3x+y)i = -1 + 13i$

$x + 3y = -1 \quad (\times 3) \qquad 3x + 9y = -3$ –

$3x + y = 13 \qquad\qquad\qquad 3x + y = 13$

$8y = -16 \Rightarrow \underline{y = -2}$

$x - 6 = -1 \Rightarrow \underline{x = 5}$

7. (a) Use the results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(2n+1)(2n-1)$$

for all positive integers n.

(6)

(b) Hence show that

$$\sum_{r=n+1}^{3n}(2r-1)^2 = \frac{2}{3}n\left(an^2+b\right)$$

where a and b are integers to be found.

(4)

$\sum(2r-1)^2 = 4\sum r^2 - 4\sum r + \sum 1$

$= 4\left(\frac{1}{6}n(n+1)(2n+1)\right) - 4\left(\frac{1}{2}n(n+1)\right) + n$

$= \frac{1}{3}n\left(2(n+1)(2n+1) - 6(n+1) + 3\right)$

$= \frac{1}{3}n\left(4n^2+6n+2-6n-6+3\right) = \frac{1}{3}n\left(4n^2-1\right)$

$= \frac{1}{3}n(2n+1)(2n-1)$ #

b) $\sum_{r=n+1}^{3n}(2r-1)^2 = \frac{1}{3}(3n)(6n+1)(6n-1) - \frac{1}{3}n(2n+1)(2n-1)$

$= \frac{1}{3}n\left(3(36n^2-1) - (4n^2-1)\right)$

$= \frac{1}{3}n\left[104n^2 - 2\right] = \frac{2}{3}n\left[52n^2-1\right]$

8. The parabola C has equation $y^2 = 48x$.

The point $P(12t^2, 24t)$ is a general point on C.

(a) Find the equation of the directrix of C.

(b) Show that the equation of the tangent to C at $P(12t^2, 24t)$ is

$$x - ty + 12t^2 = 0$$

(4)

The tangent to C at the point $(3, 12)$ meets the directrix of C at the point X.

(c) Find the coordinates of X.

(4)

a) $y^2 = 48x \equiv 4ax \Rightarrow a = 12$ directrix $x + 12 = 0$

b) $\frac{d}{dx}y^2 = \frac{d}{dx}(48x) \Rightarrow 2y\frac{dy}{dx} = 48 \Rightarrow \frac{dy}{dx} = \frac{24}{y}$

at P $m_t = \frac{24}{24t} = \frac{1}{t} \Rightarrow y - 24t = \frac{1}{t}(x - 12t^2)$

$ty - 24t^2 = x - 12t^2 \Rightarrow x - ty + 12t^2 = 0$ #

b) $(3,12) \Rightarrow 3 = 12t^2$ $t = \frac{1}{2}$ $24t = 12$ $t = \frac{1}{2}$

$t = \frac{1}{2} \rightarrow \frac{1}{2}y = x + 3 \Rightarrow y = 2x + 6$

at directrix $x = -12 \Rightarrow y = -24 + 6 = -18$ $X(-12, -18)$

9. Prove by induction, that for $n \in \mathbb{Z}^+$,

(a) $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$,

(b) $f(n) = 7^{2n-1} + 5$ is divisible by 12.

(6)

a) $n=1 \quad \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^1 = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix} \qquad n=1 \quad \begin{pmatrix} 3^1 & 0 \\ 3(3^1-1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$

$\therefore$ true for $n=1$

$n=k+1 \quad \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}$

$= \begin{pmatrix} 3(3^k) & 0 \\ 6(3^k)+3(3^k)-3 & 1 \end{pmatrix} = \begin{pmatrix} 3^{k+1} & 0 \\ 9(3^k)-3 & 1 \end{pmatrix} = \begin{pmatrix} 3^{k+1} & 0 \\ 3^{k+2}-3 & 1 \end{pmatrix}$

$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1}-1) & 1 \end{pmatrix} = \begin{pmatrix} 3^{k+1} & 0 \\ 3^{k+2}-3 & 1 \end{pmatrix} \quad \#$

$\therefore$ true for $n=1$, true for $n=k+1$ if true for $n=k$

$\therefore$ true for all $n \in \mathbb{Z}^+$ by Induction.

b) $f(1) = 7^{2-1}+5 = 12 \quad = 12 \times 1 \quad \therefore$ true for $n=1$

$f(k+1) = 7^{2(k+1)-1}+5 = 7^{2k+1}+5 = 7^{2k-1+2}+5$

$= 7^2(7^{2k-1})+5 = 49(7^{2k-1}+5)+240$

$\Rightarrow f(k+1) = 49f(k) + 20 \times 12$

$\therefore$ true for $n=1$, true for $n=k+1$ if true for $n=k$

$\therefore$ true for all $n \in \mathbb{Z}^+$ by Induction